EIGENVALUES FOR MAXWELL'S EQUATIONS WITH DISSIPATIVE BOUNDARY CONDITIONS

FERRUCCIO COLOMBINI, VESSELIN PETKOV † AND JEFFREY RAUCH

Abstract. Let $V(t) = e^{tG_b}$, $t \geq 0$, be the semigroup generated by Maxwell's equations in an exterior domain $\Omega \subset \mathbb{R}^3$ with dissipative boundary condition $E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0, \gamma(x) > 0, \forall x \in \Gamma = \partial\Omega$. We prove that if $\gamma(x)$ is nowhere equal to 1, then for every $0 < \epsilon \ll 1$ and every $N \in \mathbb{N}$ the eigenvalues of G_b lie in the region $\Lambda_\epsilon \cup \mathcal{R}_N$, where $\Lambda_\epsilon = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq C_\epsilon(|\operatorname{Im} z|^{\frac{1}{2}+\epsilon} + 1), \operatorname{Re} z < 0\}$, $\mathcal{R}_N = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq C_N(|\operatorname{Re} z| + 1)^{-N}, \operatorname{Re} z < 0\}$.

1. Introduction

Suppose that $K \subset \{x \in \mathbb{R}^3 : |x| \leq a\}$ is an open connected domain and $\Omega := \mathbb{R}^3 \setminus \bar{K}$ is an open connected domain with C^∞ smooth boundary Γ. Consider the boundary problem

$$\begin{aligned} &\partial_t E = \operatorname{curl} B, \qquad \partial_t B = -\operatorname{curl} E \quad \text{in} \quad \mathbb{R}^+_t \times \Omega, \\ &E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0 \quad \text{on} \quad \mathbb{R}^+_t \times \Gamma, \\ &E(0,x) = e_0(x), \qquad B(0,x) = b_0(x). \end{aligned} \tag{1.1}$$

with initial data $f = (e_0, b_0) \in (L^2(\Omega))^6 = \mathcal{H}$. Here $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at $x \in \Gamma$ pointing into Ω, $\langle , \rangle$ denotes the scalar product in $\mathbb{C}^3$, $u_{tan} := u - \langle u, \nu\rangle\nu$, and $\gamma(x) \in C^\infty(\Gamma)$ satisfies $\gamma(x) > 0$ for all $x \in \Gamma$. The solution of the problem (1.1) is given by a contraction semigroup $(E, B) = V(t)f = e^{tG_b}f$, $t \geq 0$, where the generator G_b has domain $D(G_b)$ that is the closure in the graph norm of functions $u = (v, w) \in (C^\infty_{(0)}(\mathbb{R}^3))^3 \times (C^\infty_{(0)}(\mathbb{R}^3))^3$ satisfying the boundary condition $v_{tan} - \gamma(\nu \wedge w_{tan}) = 0$ on Γ.

In an earlier paper [2] we proved that the spectrum of G_b in $\operatorname{Re} z < 0$ consists of isolated eigenvalues with finite multiplicity. If $G_b f = \lambda f$ with $\operatorname{Re}\lambda < 0$, the solution $u(t,x) = V(t)f = e^{\lambda t}f(x)$ of (1.1) has exponentially decreasing global energy. Such solutions are called **asymptotically disappearing** and they are invisible for inverse scattering problems. It was proved [2] that if there is at least one eigenvalue λ of G_b with $\operatorname{Re}\lambda < 0$, then the wave operators $W_\pm$ are not complete, that is $\operatorname{R}an\, W_- \neq \operatorname{R}an\, W_+$. Hence we cannot define the scattering operator S related to the Cauchy problem for the Maxwell system and (1.1) by the product $W_+^{-1}W_-$. For the perfect conductor boundary conditions for Maxwell's equations, the energy is conserved in time and the unperturbed and perturbed problems are associated to unitary groups. The corresponding scattering operator $S(z) : (L^2(\mathbb{S}^2))^2 \to (L^2(\mathbb{S}^2))^2$ satisfies the identity

$$S^{-1}(z) = S^*(\bar{z}), \quad z \in \mathbb{C} \tag{1.2}$$

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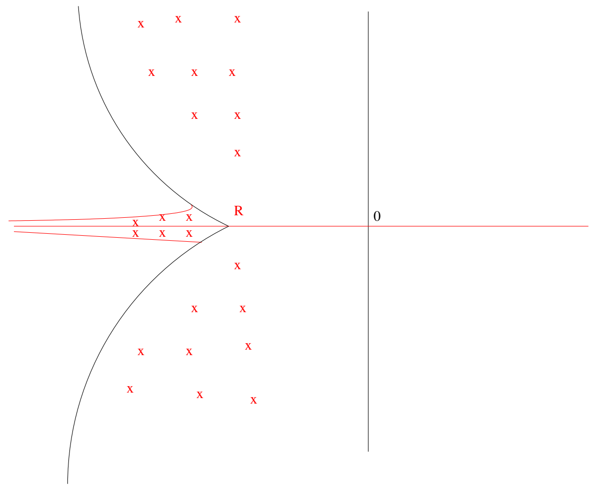

FIGURE 1. Eigenvalues of G_b

if $S(z)$ is invertible at z. The scattering operator $S(z)$ defined in [5] is such that $S(z)$ and $S^*(z)$ are analytic in the "physical" half plane $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ and the above relation for conservative boundary conditions implies that $S(z)$ is invertible for $\operatorname{Im} z > 0$. For dissipative boundary conditions the relation (1.2) in general is not true and $S(z_0)$ may have a non trivial kernel for some $z_0, \operatorname{Im} z_0 > 0$. Lax and Phillips [5] proved that this implies that $\mathbf{i}z_0$ is an eigenvalue of G_b. The analysis of the location of the eigenvalues of G_b is important for the location of the points where the kernel of $S(z)$ is not trivial.

The main result of this paper is the following (see Figure 1)

Theorem 1.1. *Assume that for all $x \in \Gamma$, $\gamma(x) \neq 1$. Then for every $0 < \epsilon \ll 1$ and every $N \in \mathbb{N}$ there are constants $C_\epsilon > 0$ and $C_N > 0$ such that the eigenvalues of G_b lie in the region $\Lambda_\epsilon \cup \mathcal{R}_N$, where*

$$\Lambda_\epsilon = \{z \in \mathbb{C} : \ |\operatorname{Re} z| \leq C_\epsilon(|\operatorname{Im} z|^{1/2+\epsilon} + 1), \ \operatorname{Re} z < 0\},$$
$$\mathcal{R}_N = \{z \in \mathbb{C} : \ |\operatorname{Im} z| \leq C_N(|\operatorname{Re} z| + 1)^{-N}, \ \operatorname{Re} z < 0\}.$$

If $\operatorname{Re} \lambda < 0$ and $G_b(E, B) = \lambda(E, B) \neq 0$, then

$$\begin{aligned} \lambda E &= \operatorname{curl} B && \text{on} \quad \Omega, \\ \lambda B &= -\operatorname{curl} E && \text{on} \quad \Omega, \\ \operatorname{div} E &= \operatorname{div} B = 0, && \text{on} \quad \Omega, \\ E_{tan} - \gamma(\nu \wedge B_{tan}) &= 0 && \text{on} \quad \Gamma. \end{aligned} \tag{1.3}$$

This implies that $u := (E, B)$ satisfies

$$\Delta u - \lambda^2 u = 0, \qquad \text{on} \quad \Omega.$$

The eigenvalues of G_b are symmetric with respect to the real axis, so it is sufficient to examine the location of the eigenvalues whose imaginary part is nonnegative. The mapping $z \mapsto z^2$ maps the positive quadrant $\{z \in \mathbb{C} : \operatorname{Re} z > 0\,, \operatorname{Im} z > 0\}$ bijectively to the upper half space. Denote by $\sqrt{z}$ the inverse map. The part of the

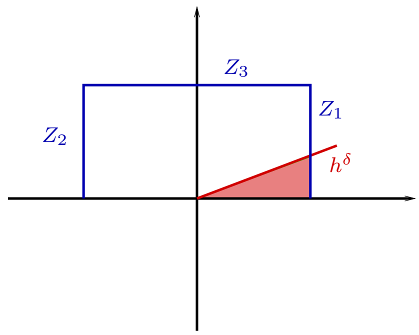

FIGURE 2. Contours $Z_1, Z_2, Z_3, \delta = 1/2 - \epsilon$

spectral domain $\{\lambda \in \mathbb{C} :\ \mathrm{Re}\,\lambda < 0\,,\ \mathrm{Im}\,\lambda > 0\}$ is mapped by $\lambda = \mathbf{i}\sqrt{z}$ to the upper half plane $\{z \in \mathbb{C} :\ \mathrm{Im}\, z > 0\}$. In $\{z \in \mathbb{C} :\ \mathrm{Im}\, z \geq 0\}$ introduce the sets

$$\begin{aligned} Z_1 &:= \{z \in \mathbb{C} :\ \mathrm{Re}\, z = 1,\ \ h^\delta \leq \mathrm{Im}\, z \leq 1\}, \quad 0 < h \ll 1, \quad 0 < \delta < 1/2, \\ Z_2 &:= \{z \in \mathbb{C} : \mathrm{Re}\, z = -1,\ \ 0 \leq \mathrm{Im}\, z \leq 1\}, \\ Z_3 &:= \{z \in \mathbb{C} : |\,\mathrm{Re}\, z| \leq 1,\ \ \mathrm{Im}\, z = 1\}. \end{aligned}$$

Set $\lambda = \mathbf{i}\sqrt{z}/h,\ z \in Z_1 \cup Z_2 \cup Z_3$. To study the eigenvalues $\lambda,\ |\lambda| > R_0$, it is sufficient to consider $0 < h \ll 1$. As z runs over the rectangle in Figure 2, with $0 < h \ll 1$, λ sweeps out the large values in the intersection of left and upper half planes. The values of $z \in Z_2$ near the lower left hand corner, $z = -1$, of the rectangle go the spectral values near the negative real axis. The spectral analysis near these values in Z_2 for dissipative Maxwell's equations does not have clear analogue with the spectral problems for the wave equation with dissipative boundary conditions. In fact, for the wave equation if $0 < \gamma(x) < 1,\ \forall x \in \Gamma$, the eigenvalues of the generator of the corresponding semigroup are located in the domain Λ_ϵ (see Section 3, [8] and [6]). For Maxwell's equations the eigenvalues of G_b lie in the domain $\Lambda_\epsilon \cup \mathcal{R}_N$ and for $0 < \gamma(x) < 1$ and $\gamma(x) > 1$ we have the same location (see Appendix for the case $K = \{x \in \mathbb{R}^3 :\ \|x| \leq 1\}$).

Equation (1.3) implies that on Ω each eigenfunction $u = (E, B)$ of G_b satisfies

$$\sqrt{z}\, E \;=\; \frac{h}{\mathbf{i}}\mathrm{curl}\, B\,, \qquad \sqrt{z}\, B \;=\; -\frac{h}{\mathbf{i}}\mathrm{curl}\, E\,, \tag{1.4}$$

and therefore $(-h^2\Delta - z)E = (-h^2\Delta - z)B = 0$. For eigenfunctions $(E, B) \neq 0$, we derive a pseudodifferential system on the boundary involving $E_{tan} = E - \langle E, \nu\rangle\nu$ and $E_{nor} = \langle E, \nu\rangle$. A semi-classical analysis shows that for $z \in Z_1 \cup Z_3$ this system implies that for h small enough we have $E|_\Gamma = 0$ which yields $E = B = 0$. By scaling one concludes that the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G_b lie in the region $\Lambda_\epsilon \cup \mathcal{M}$, where

$$\mathcal{M} = \{z \in \mathbb{C} :\ |\arg z - \pi| \leq \pi/4,\ |z| \geq R_0 > 0,\ \mathrm{Re}\, z < 0\}.$$

The strategy for the analysis of the case $z \in Z_1 \cup Z_3$ is similar to that exploited in [9] and [8]. In these papers the semi-classical Dirichlet-to-Neumann map $\mathcal{N}(z, h)$ plays a crucial role and the problem is reduced to the proof that some

$h-$pseudodifferantial operators is elliptic in a suitable class. For the Maxwell system the pseudodifferential equation on the boundary is more complicated. Using the equation $\operatorname{div} E = 0$, yields a pseudodifferential system for E_{tan} and E_{nor}. We show that if $(E, B) \neq 0$ is an eigenfunction of G_b, then $\|E_{nor}\|_{H_h^1(\Gamma)}$ is bounded by $Ch\|E_{tan}\|_{H_h^1(\Gamma)}$. The term involving E_{nor} then plays the role of a negligible perturbation in the pseudodifferentrial system on the boundary and this reduces the analysis to one involving only E_{tan}. The system concerning E_{tan} has a diagonal leading term and we may apply the same arguments as those of [8] to conclude that $E_{tan} = 0$ and hence $E_{nor} = 0$.

The analysis of the case $z \in Z_2$ is more difficult since the principal symbol g of the pseudodifferential system for E_{tan} need not be elliptic at some points (see Section 3). Even where g is elliptic, if $|\operatorname{Im} z| \leq h^{1/2}$ it is difficult to estimate the norm of the difference $Op_h(g)Op_h(g^{-1}) - I$. To show that the eigenvalues of G_b lying in $\mathcal{M}$ are in fact confined to the region $\mathcal{R}_N$ for every $N \in \mathbb{N}$, we analyze the real part of the following scalar product in $L^2(\Gamma)$

$$Q(E_0) \ := \ \operatorname{Re}\langle(\mathcal{N}(z,h) - \sqrt{z}\gamma)E_0, E_0\rangle_{L^2(\Gamma)}, \qquad E_0 \ := \ E|_{\Gamma}.$$

We follow the approach in [9], [8] based on a Taylor expansion of $Q(E_0)$ at $z = -1$ and the fact that for $z = -1$ we have $Q(E_0) = \mathcal{O}(h^N)$, $\forall N \in \mathbb{N}$. In the Appendix we treat the case when $K = \{x \in \mathbb{R}^3 : \ |x| \leq 1\}$ is a ball and $\gamma = \text{const}$. We prove that for $\gamma \equiv 1$ the operator G has no eigenvalues in $\{\operatorname{Re} z < 0\}$, while for every $\gamma \in \mathbb{R}^+ \setminus \{1\}$ we have infinite number of real eigenvalues.

2. Pseudodifferential equation on the boundary

Introduce geodesic normal coordinates $(y_1, y') \in \mathbb{R}^3$ on a neighborhood of a point $x_0 \in \Gamma$ as follows. For a point x, $y'(x)$ is the closest point in Γ and $y_1 = \operatorname{dist}(x, \Gamma)$. Define $\nu(x)$ to be the unit normal in the direction of increasing y_1 to the surface $y_1 = \text{constant}$ through x. Thus $\nu(x)$ is an extension of the unit normal vector to a unit vector field. The boundary Γ is mapped to $y_1 = 0$ and

$$x \ = \ \alpha(y_1, y') \ = \ \beta(y') \ + \ y_1\nu(y').$$

We have

$$\frac{\partial}{\partial x_k} \ = \ \nu_k(y')\frac{\partial}{\partial y_1} + \sum_{j=2}^{3} \frac{\partial y_j}{\partial x_k}\frac{\partial}{\partial y_j}, \qquad k = 1,2,3.$$

Moreover,

$$\sum_{k=1}^{3} \nu_k(y')\frac{\partial y_j}{\partial x_k}(y_1, y') = \langle \nu, \frac{\partial y_j}{\partial x}\rangle = 0, \qquad j = 1,2,3, \quad \text{and}$$

$$\sum_{k=1}^{3} \nu_k(x)\partial_{x_k} f(x) = \partial_{y_1}(f(\alpha(y_1, y')).$$

Since $\|\nu(x)\| = 1$, $\langle \nu, \partial_{x_j}\nu\rangle = 0$, $j = 1,2,3.$

A straight forward computation yields

$$\begin{aligned}\nu(x) \wedge \frac{h}{\mathbf{i}}\operatorname{curl} u(x) &= \mathbf{i}h\partial_\nu u_{tan} + \Big(\langle D_{x_1}u, \nu\rangle, \langle D_{x_2}u, \nu\rangle, \langle D_{x_3}u, \nu\rangle\Big)\Big|_{tan} \\ &= \mathbf{i}h\partial_\nu u_{tan} + \Big(\operatorname{grad}_h \langle u, \nu\rangle\Big)\Big|_{tan} - \mathbf{i}hg_0(u_{tan}), \qquad x \in \Gamma,\end{aligned}$$

where

$$D_{x_j} = -\mathbf{i}h\partial_{x_j}, \qquad j = 1,2,3, \quad \mathrm{grad}_{\,h} f = \{D_{x_j} f\}_{j=1,2,3},$$
$$g_0(u_{tan}) = \{\langle u_{tan}, \partial_{x_j}\nu\rangle\}_{j=1,2,3}.$$

Setting $E_{nor} = \langle E, \nu\rangle$, from (1.3) one deduces

$$\nu \wedge B = -\frac{1}{\sqrt{z}}\nu \wedge \frac{h}{\mathbf{i}}\mathrm{curl}\, E = \frac{1}{\sqrt{z}} D_\nu E_{tan} - \frac{1}{\sqrt{z}}\Big[\Big(\mathrm{grad}_{\,h} E_{nor}\Big)\Big|_{tan} - \mathbf{i}hg_0(E_{tan})\Big],$$

where $D_\nu = -\mathbf{i}h\partial_\nu$ and the boundary condition in (1.3) becomes

$$\Big(D_\nu - \frac{1}{\gamma}\sqrt{z}\Big)E_{tan} - \Big(\mathrm{grad}_{\,h} E_{nor}\Big)\Big|_{tan} + \mathbf{i}hg_0(E_{tan}) = 0, \qquad x \in \Gamma. \tag{2.1}$$

Next

$$\mathrm{grad}_{\,h} f(x)|_{tan} \;=\; \Big\{\sum_{j=2}^{3} \frac{\partial y_j}{\partial_{x_k}} D_{y_j} f(\alpha(y_1, y'))\Big\}_{k=1,2,3}$$

and for $u = (u_1, u_2, u_3) \in \mathbb{C}^3$,

$$\frac{h}{\mathbf{i}}\mathrm{div}\, u(\alpha(y_1, y')) = \langle D_{y_1} u(\alpha(y_1,y')), \nu(y')\rangle + \sum_{k=1}^{3}\sum_{j=2}^{3} \frac{\partial y_j}{\partial_{x_k}} D_{y_j} u_k(\alpha(y_1, y'))$$

$$= D_{y_1}\Big(u_{nor}(y_1, y')\Big) + \sum_{j=2}^{3} D_{y_j}\Big\langle u_{tan}(\alpha(y_1,y')), \frac{\partial y_j}{\partial x}\Big\rangle + h\langle u_{tan}, Z\rangle,$$

where $\langle u(\alpha(y_1,y')), \nu(y')\rangle := u_{nor}(y_1, y')$ and Z depends on the second derivatives of $y_j,\ j = 2.3$. Apply the operator $D_{y_1} - \frac{\sqrt{z}}{\gamma(y')}$ to $\mathrm{div}\, E(\alpha(y_1, y')) = 0$ to find

$$(D_{y_1}^2 - \frac{\sqrt{z}}{\gamma(y')} D_{y_1}) E_{nor}(y_1, y') + \sum_{j=2}^{3} D_{y_j}\Big\langle (D_{y_1} - \frac{\sqrt{z}}{\gamma(y')}) E_{tan}(\alpha(y_1, y')), \frac{\partial y_j}{\partial x}\Big\rangle$$

$$= h\langle (D_{y_1} - \frac{\sqrt{z}}{\gamma}) E_{tan}, Z\rangle + h\langle E_{tan}, Z_1\rangle,$$

where $\gamma(y') := \gamma(\beta(y'))$.

Taking the trace $y_1 = 0$ and applying the boundary condition (2.1), yields

$$\Big(D_{y_1}^2 + \sum_{j,\mu=2}^{3}\sum_{k=1}^{3} \frac{\partial y_j}{\partial x_k}\frac{\partial y_\mu}{\partial x_k} D^2_{y_j, y_\mu}\Big) E_{nor}(0, y') - \frac{\sqrt{z}}{\gamma(y')} D_{y_1} E_{nor}(0, y')$$
$$= h\Big\langle \Big(\mathrm{grad}_{\,h} E_{nor}\Big)\Big|_{tan}(0, y'), Z\Big\rangle + hQ_1(E_{tan}(0, y')), \tag{2.2}$$

with

$$\|Q_1(E_{tan}(0, y'))\|_{L^2(\mathbb{R}^2)} \le C_2 \|E_{tan}(0, y')\|_{H_h^1(\mathbb{R}^2)}.$$

Here $H_h^s(\Gamma), s \in \mathbb{R}$, denotes the semi-classical Sobolev spaces with norm $\|\langle h\partial_x\rangle^s u\|_{L^2(\Gamma)}$, $\langle h\partial_x\rangle = (1 + \|h\partial_x\|^2)^{1/2}$. In the exposition below we use the spaces $(L^2(\Gamma))^3$ and $(H_h^s(\Gamma))^3$ of vector-valued functions but we will omit this in the notations writing simply $L^2(\Gamma)$ and $H_h^s(\Gamma)$.

The operator $-h^2\Delta_x - z$ in the coordinates (y_1, y') has the form

$$\mathcal{P}(z, h) = D_{y_1}^2 + r(y, D_{y'}) + q_1(y, D_y) + h^2\tilde{q} - z$$

with $r(y,\eta') = \langle R(y)\eta', \eta'\rangle$, $q_1(y,\eta) = \langle q_1(y), \eta)\rangle$. Here

$$R(y) = \Big\{\sum_{k=1}^{3} \frac{\partial y_j}{\partial x_k}\frac{\partial y_\mu}{\partial x_k}\Big\}_{j,\mu=2}^{3} = \Big\{\Big\langle \frac{\partial y_j}{\partial x}, \frac{\partial y_\mu}{\partial x}\Big\rangle\Big\}_{j,\mu=2}^{3}$$

is a symmetric (2×2) matrix and $r(0,y',\eta') = r_0(y',\eta')$, where $r_0(y',\eta')$ is the principal symbol of the Laplace-Beltrami operator $-h^2\Delta_\Gamma$ on Γ equipped with the Riemannian metric induced by the Euclidean one in $\mathbb{R}^3$. We have

$$\Big(\mathcal{P}(z,h)E_{nor}\Big)(0,y') \; = \; \langle \mathcal{P}(z,h)E, \nu\rangle(0,y') \; + \; hQ_2(E(0,y')),$$

where

$$\|Q_2(E(0,y'))\|_{L^2(\mathbb{R}^2)} \; \leq \; C_2\, \|E(0,y')\|_{H_h^1(\mathbb{R}^2)}.$$

Since $\mathcal{P}(z,h)E = 0$, this lets us replace the terms with all second derivatives of E_{nor} in (2.4) by $zE_{nor}(0,y')$ modulo terms having a factor h and containing first order derivatives of E_{nor}. This follows from the form of the matrix $R(y)$ given above. After a multiplication by $-\frac{\gamma(y')}{\sqrt{z}}$ the equation (2.2) yields

$$(D_{y_1} - \gamma(y')\sqrt{z})E_{nor}(0,y') = hQ_3(E(0,y')), \tag{2.3}$$

where $Q_3(E(0,y'))$ has the same properties as $Q_2(E(0,y'))$.

Let $\psi(x) \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with support in small neighborhood of $x_0 \in \Gamma$. Replace E, B by $E_\psi = E\psi$, $B_\psi = B\psi$. The above analysis works for E_ψ and B_ψ with lower order terms depending on ψ. We obtain

$$\langle (D_\nu - \gamma(x)\sqrt{z})E|_\Gamma \psi(x), \nu(x)\rangle \; = \; h\, Q_{3,\psi}(E|_\Gamma).$$

Taking a partition of unity in a neighborhood of Γ, yields

$$\langle (D_\nu - \gamma(x)\sqrt{z})E|_\Gamma, \nu\rangle = hQ_4(E|_\Gamma), \qquad \|Q_4(E|_\Gamma)\|_{L^2(\Gamma)} \leq C\|E|_\Gamma\|_{H_h^1(\Gamma)}. \tag{2.4}$$

For $z \in Z_1 \cup Z_2 \cup Z_3$ let $\rho(x',\xi',z) = \sqrt{z - r_0(x',\xi')} \in C^\infty(T^*\Gamma)$ be the root of the equation

$$\rho^2 + r_0(x',\xi') - z = 0$$

with $\operatorname{Im}\rho(x',\xi',z) > 0$. For large $|\xi'|$,

$$\rho(x',\xi',z) \; \sim \; |\xi'|, \qquad \operatorname{Im}\rho(x',\xi',z) \; \sim \; |\xi'|,$$

while for bounded $|\xi'|$,

$$\operatorname{Im}\rho(x',\xi',z) \; \geq \; \frac{h^\delta}{C}.$$

We recall some basic facts about h-pseudodifferential operators that the reader can find in [3]. Let X be a C^∞ smooth compact manifold without boundary with dimension $d \geq 2$. Let (x,ξ) be the coordinates in $T^*(X)$ and let $a(x,\xi,h) \in C^\infty(T^*(X))$. Given $m \in \mathbb{R}$, $l \in \mathbb{R}, \delta > 0$ and a function $c(h) > 0$, one denotes by $S_\delta^{l,m}(c(h))$ the set of symbols so that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi,h)| \leq C_{\alpha,\beta}(c(h))^{-l-\delta(|\alpha|+|\beta|)}(1+|\xi|)^{m-|\beta|},\ \forall \alpha, \forall \beta, \quad (x,\xi) \in T^*(X).$$

If $c(h) = h$, we denote $S_\delta^{l,m}(c(h))$ simply by $S_\delta^{l,m}$. Symbols restricted to a domain where $|\xi| \leq C$ will be denoted by $a \in S_\delta^l(c(h))$. The $h-$pseudodifferential operator

with symbol $a(x,\xi,h)$ acts by

$$(Op_h(a)f)(x) \ := \ (2\pi h)^{-d+1}\int_{T^*X} e^{-\mathbf{i}\langle x-y,\xi\rangle/h}a(x,\xi,h)f(y)dyd\xi.$$

For matrix valued symbols we use the same definition. This means that every element of a matrix symbol is in the class $S^{l,m}_{\delta}(c(h))$.
Now suppose that $a(x,\xi,h)$ satisfies the estimates

$$|\partial_x^{\alpha}a(x,\xi,h)| \leq c_0(h)h^{-|\alpha|/2}, \qquad (x,\xi)\in T^*(X) \tag{2.5}$$

for $|\alpha|\leq d-1$, where $c_0(h)>0$ is a parameter. Then there exists a constant $C>0$ independent of h such that

$$\|Op_h(a)\|_{L^2(X)\to L^2(X)} \ \leq \ C\,c_0(h). \tag{2.6}$$

For $0\leq\delta<1/2$ products of h-pseudodifferential operators are well behaved. If $a\in S^{l_1,m_1}_{\delta}$, $b\in S^{l_2,m_2}_{\delta}$ and $s\in\mathbb{R}$, then

$$\|Op_h(a)Op_h(b)-Op_h(ab)\|_{H^s(X)\to H^{s-m_1-m_2+1}(X)}\leq Ch^{-l_1-l_2-2\delta+1}. \tag{2.7}$$

Let $u\in\mathbb{C}^3$ be the solution of the Dirichlet problem

$$(-h^2\Delta-z)u=0 \quad \text{on} \quad \Omega, \qquad u=F \quad \text{on} \quad \Gamma. \tag{2.8}$$

Introduce the semi-classical Dirichlet-to-Neumann map

$$\mathcal{N}(z,h): H_h^s(\Gamma)\ni F\longrightarrow D_{\nu}u|_{\Gamma}\in H_h^{s-1}(\Gamma).$$

G. Vodev [9] established for bounded domains $K\subset\mathbb{R}^d, d\geq 2$, with C^{∞} boundary the following approximation of the interior Dirichlet-to-Neumann map $\mathcal{N}_{int}(z,h)$ related to (2.8), where the equation $(-h^2\Delta-z)u=0$ is satisfied in K.

Theorem 2.1 ([9]). *For every $0<\epsilon\ll 1$ there exists $0<h_0(\epsilon)\ll 1$ such that for $z\in Z_{1,\epsilon}:=\{z\in Z_1,\ |\operatorname{Im} z|\geq h^{\frac{1}{2}-\epsilon}\}$ and $0<h\leq h_0(\epsilon)$ we have*

$$\|\mathcal{N}_{int}(z,h)(F)-Op_h(\rho+hb)F\|_{H_h^1(\Gamma)}\leq\frac{Ch}{\sqrt{|\operatorname{Im} z|}}\|F\|_{L^2(\Gamma)}, \tag{2.9}$$

where $b\in S^0_{0,1}(\Gamma)$ does not depend on h and z. Moreover, (2.9) holds for $z\in Z_2\cup Z_3$ with $|\operatorname{Im} z|$ replaced by 1.

With small modifications (2.9) holds for the Dirichlet-to-Neumann map $\mathcal{N}(z,h)$ related to (2.8) (see [8]). Applying (2.9) with $\mathcal{N}(z,h)$ and $F=E_0=E|_{\Gamma}$, we obtain

$$\Big\|\langle\mathcal{N}(z,h)E_0,\nu\rangle-\langle Op_h(\rho)E_0,\nu\rangle\Big\|_{L^2(\Gamma)}\leq\frac{Ch}{\sqrt{|\operatorname{Im} z|}}\|E_0\|_{L^2(\Gamma)}. \tag{2.10}$$

Therefore (2.4) yields

$$\Big\|\langle Op_h(\rho)-\gamma\sqrt{z})E_0,\nu\rangle-hQ_4(E_0)\Big\|_{L^2(\Gamma)}\leq\frac{Ch}{\sqrt{|\operatorname{Im} z|}}\|E_0\|_{L^2(\Gamma)}. \tag{2.11}$$

The commutator $[Op_h(\rho),\nu(x)]$ is a pseudodifferential operator with symbol in $h^{1-\delta}S^{0,0}_{\delta}$ and so

$$\|[Op_h(\rho),\nu_k(x)]E_{nor}\|_{H_h^j(\Gamma)} \ \leq \ C_2h^{1-\delta}\|E_{nor}\|_{H_h^j(\Gamma)}, \quad k=1,2,3, \quad j=0,1.$$

The last estimate combined with (2.11) implies

$$\Big\|(Op_h(\rho)-\gamma\sqrt{z})E_{nor}-hQ_4(E_0)\Big\|_{L^2(\Gamma)}\leq C_3\Big(\frac{h}{\sqrt{|\operatorname{Im} z|}}+h^{1-\delta}\Big)\|E_0\|_{L^2(\Gamma)}. \tag{2.12}$$

3. Eigenvalues-free regions

For $z \in Z_{1,\epsilon}$ we have $\rho \in S^{0,1}_{\delta}$ with $0 < \delta = 1/2 - \epsilon < 1/2$, while for $z \in Z_2 \cup Z_3$ we have $\rho \in S^{0,1}_0$ (see [9]). Since Γ is connected one has either $\gamma(x) > 1$ or $0 < \gamma(z) < 1$. We present the analysis in the case where $0 < \gamma(x) < 1$, $\forall x \in \Gamma$. The case $1 < \gamma(x)$ is reduced to this case at the end of the section. Clearly, there exists $\epsilon_0 > 0$ such that

$$\epsilon_0 \leq \gamma(x) \leq 1 - \epsilon_0, \qquad \forall x \in \Gamma.$$

Combing (2.4) and (2.9), yields

$$\|\langle (Op_h(\rho) - \gamma(x)\sqrt{z})E_0, \nu(x)\rangle\|_{L^2(\Gamma)} \leq C\frac{h}{\sqrt{|\operatorname{Im} z|}}\|E_0\|_{L^2(\Gamma)} + C_1 h\|E_0\|_{H^1_h(\Gamma)},$$

where for $z \in Z_2 \cup Z_3$ we can replace $|\operatorname{Im} z|$ by 1. This estimate for E_0 and the estimate for the commutator $[Op_h(\rho), \nu_k(x)]$ imply

$$\|(Op_h(\rho) - \gamma(x)\sqrt{z})E_{nor}\|_{L^2(\Gamma)} \leq \frac{C_3\, h}{\sqrt{|\operatorname{Im} z|}}\|E_0\|_{L^2(\Gamma)} + C_4 h^{1-\delta}\|E_0\|_{H^1_h(\Gamma)}. \quad (3.1)$$

Let (x', ξ') be coordinates on $T^*(\Gamma)$. Consider the symbol

$$c(x', \xi', z) := \ \rho(x', \xi', z) \ - \ \gamma(x')\sqrt{z}, \qquad x' \in \Gamma.$$

Following the analysis in Section 3, [8], we know that c is elliptic in the case $0 < \gamma(x') < 1$ and if $z \in Z_1$ we have $c \in S^{0,1}_{\delta}$, $|\operatorname{Im} z| c^{-1} \in S^{0,-1}_{\delta}$, while if $z \in Z_2 \cup Z_3$ one gets $c \in S^{0,1}_0$, $c^{-1} \in S^{0,-1}_0$. This implies

$$\|Op_h(c^{-1})Op_h(c)E_{nor}\|_{H^1_h(\Gamma)} \ \leq \ \frac{C}{|\operatorname{Im} z|}\|Op_h(c)E_{nor}\|_{L^2(\Gamma)}.$$

On the other hand, according to Section 7 in [3], the symbol of the operator $Op_h(c^{-1})Op_h(c) - I$ is given by

$$\sum_{j=1}^{N}\frac{(ih)^j}{j!}\sum_{|\alpha|=j} D^{\alpha}_{\xi'}(c^{-1})(x', \xi')D^{\alpha}_{y'}c(y', \eta')\big|_{x'=y', \xi'=\eta'} + \tilde{b}_N(x', \xi')$$

$$:= b_N(x', \xi') + \tilde{b}_N(x', \xi'),$$

where

$$|\partial^{\alpha}_{x'}\tilde{b}_N(x', \xi')| \ \leq \ C_{\alpha} h^{N(1-2\delta) - s_d - |\alpha|/2}.$$

Taking into account the estimates for c^{-1} and c, and applying (2.5), and (2.6) yields

$$\Big\|\Big(Op_h(c^{-1})Op_h(c) - I\Big)E_{nor}\Big\|_{H^j_h(\Gamma)} \leq C_5\frac{h}{|\operatorname{Im} z|^2}\|E_{nor}\|_{H^j_h(\Gamma)}, \quad j = 0, 1.$$

Repeating the argument in Section 3 in [8] concerning the case $0 < \gamma(x') < 1$, for $z \in Z_1$ and $0 < \delta < 1/2$, one finds

$$\|E_{nor}\|_{H^1_h(\Gamma)} \leq \Big\|\Big(Op_h(c^{-1})Op_h(c) - I\Big)E_{nor}\Big\|_{H^1_h(\Gamma)} + \Big\|Op_h(c^{-1})Op_h(c)E_{nor}\Big\|_{H^1_h(\Gamma)}$$
$$\leq C_6 h^{1-2\delta}\|E_0\|_{L^2(\Gamma)} + C_5 h^{1-2\delta}\|E_{nor}\|_{H^1_h(\Gamma)} + C_7 h^{1-\delta}\|E_0\|_{H^1_h(\Gamma)}. \quad (3.2)$$

Clearly,

$$\|E_0\|_{H^k_h(\Gamma)} \leq \|E_{tan}\|_{H^k_h(\Gamma)} + B_k\|E_{nor}\|_{H^k_h(\Gamma)}, \qquad k \in \mathbb{N}$$

with B_k independent of h. Hence we can absorb the terms involving the norms of E_{nor} in the right hand side of (3.2) choosing h small enough, and we get

$$\|E_{nor}\|_{H_h^1(\Gamma)} \leq Ch^{1-2\delta}\|E_{tan}\|_{H_h^1(\Gamma)}. \tag{3.3}$$

The analysis of the case $z \in Z_2 \cup Z_3$ is simpler since in the estimates above we have no coefficient $|\operatorname{Im} z|^{-1}$ and we obtain the same result with a factor h on the right hand side of (3.3).

With a similar argument it is easy to show that

$$\|E_{nor}\|_{L^2(\Gamma)} \ \leq \ C'h^{1-2\delta}\|E_{tan}\|_{L^2(\Gamma)}. \tag{3.4}$$

In fact from (2.12) one obtains

$$\Big\|Op_h(c^{-1})\Big[(Op_h(\rho)-\gamma\sqrt{z})E_{nor}-hQ_4(E_0)\Big]\Big\|_{L^2(\Gamma)} \leq \frac{C_8}{|\operatorname{Im} z|}\Big(\frac{h}{\sqrt{|\operatorname{Im} z|}}+h^{1-\delta}\|E_0\|_{L^2(\Gamma)}\Big)$$

and

$$\|Op_h(c^{-1})Q_4(E_0)\|_{L^2(\Gamma)} \leq \frac{C_9}{|\operatorname{Im} z|}\|E_0\|_{L^2(\Gamma)}.$$

Combining these estimates with the estimate of $\|Op_h(c^{-1})Op_h(c) - I\|_{L^2(\Gamma)\to L^2(\Gamma)}$ yields (3.4).

Going back to the equation (2.1), we have

$$\begin{aligned}\Big(D_\nu - \frac{1}{\gamma}\sqrt{z}\Big)E = \Big(D_\nu - \gamma\sqrt{z}\Big)E_{nor}\nu - (\frac{1}{\gamma} - \gamma)\sqrt{z}E_{nor}\nu \\ +\mathbf{i}hg_0(E_{tan}) + \Big(\operatorname{grad}_h(E_{nor})\Big)\Big|_{\tan}, \qquad x \in \Gamma.\end{aligned} \tag{3.5}$$

Notice that for the first term on the right hand side of (3.5) we can apply the equality (2.4), while for E_{nor} and $\Big(\operatorname{grad}_h(E_{nor})\Big)\Big|_{\tan}$ we have a control by the estimate (3.3). Consequently, setting $E_0 = E|_\Gamma$, the right hand side of (3.5) is bounded by $Ch^{1-2\delta}\|E_0\|_{H_h^1(\Gamma)}$. Next

$$1 \ < \ \frac{1}{1-\epsilon_0} \ \leq \ \frac{1}{\gamma(x)} \ \leq \ \frac{1}{\epsilon_0}, \qquad \forall x \in \Gamma.$$

This corresponds to the case (B) examined in Section 4 of [8]. The approximation of the operator $\mathcal{N}(z,h)$ given by (2.9) yields the estimate

$$\|(Op_h(\rho) - \frac{1}{\gamma}\sqrt{z})E_0\|_{L^2(\Gamma)} \ \leq \ C\Big(\frac{h}{\sqrt{|\operatorname{Im} z|}}\|E_0\|_{L^2(\Gamma)} + h^{1-2\delta}\|E_0\|_{H_h^1(\Gamma)}\Big). \tag{3.6}$$

For $z \in Z_1 \cup Z_3$ the symbol

$$d(x',\xi',z) := \rho(x',\xi',z) - \frac{1}{\gamma(x')}\sqrt{z}$$

is elliptic (see Section 4, [8]) and $d \in S_\delta^{0,1}$, $d^{-1} \in S_\delta^{0,-1}$. Then from (3.6) we estimate $\|E_0\|_{H_h^1(\Gamma)}$ and we obtain $E_0 = 0$ for h small enough. This implies $E = B = 0$.

Now recall that we have

$$\operatorname{Re}\lambda = -\frac{\operatorname{Im}\sqrt{z}}{h},\ \operatorname{Im}\lambda = \frac{\operatorname{Re}\sqrt{z}}{h}.$$

Suppose that $z \in Z_1$. Then

$$|\operatorname{Re}\lambda| \geq C(h^{-1})^{1-\delta},\ |\operatorname{Im}\lambda| \leq C_1 h^{-1} \leq C_2 |\operatorname{Re}\lambda|^{\frac{1}{1-\delta}}.$$

So if

$$|\operatorname{Re}\lambda| \geq C_3 |\operatorname{Im}\lambda|^{1-\delta},\ \operatorname{Re}\lambda \leq -C_4 < 0,$$

there are no eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G_b. In the same way we handle the case $z \in Z_3$ and we conclude that if $z \in Z_1 \cup Z_3$ for every $\epsilon > 0$ the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G_b lie in the domain $\Lambda_\epsilon \cup \mathcal{M}$, where

$$\mathcal{M} = \{z \in \mathbb{C} :\ |\arg z - \pi| \leq \pi/4,\ |z| \geq R_0 > 0,\ \operatorname{Re} z < 0\},$$

Λ_ϵ being the domain introduced in Theorem 1.1. Of course, if we consider the domain

$$Z_{3,\delta_0} = \{z \in \mathbb{C} :\ |\operatorname{Re} z| \leq 1,\quad \operatorname{Im} z = \delta_0 > 0\},$$

instead of Z_3, we obtain an eigenvalue-free region with $\mathcal{M}$ replaced by

$$\mathcal{M}_{\delta_0} = \{z \in \mathbb{C} :\ |\arg z - \pi| \leq \mathrm{a}rctg\, \delta_0,\ |z| \geq R_0(\delta_0) > 0,\ \operatorname{Re} z < 0\}.$$

The investigation of the case $z \in Z_2$ is more complicated since the symbol d may vanish for $\operatorname{Im} z = 0$ and $(x_0', \xi_0') \in T^*(\Gamma)$ satisfying the equation

$$\sqrt{1 + r_0(x_0', \xi_0')} - \frac{1}{\gamma(x_0')} \ =\ 0.$$

To cover this case and to prove that the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ with $z \in Z_2$ are confined in the domain $\mathcal{R}_N$, $\forall N \in \mathbb{N}$, we follow the arguments in [9] and [8]. For $z \in Z_2$ we introduce an operator $T(z,h)$ that yields a better approximation of $\mathcal{N}(z,h)$. In fact, $T(z,h)$ is defined by the construction of the semi-classical parametrix in Section 3, [9] for the problem (2.8) with $F = E_0$. We refer to [9] for the precise definition of $T(z,h)$ and more details. For our exposition we need the next proposition. Since $(\Delta - z)E = 0$, as in [9], we obtain

Proposition 3.1. *For $z \in Z_2$ and every $N \in \mathbb{N}$ we have the estimate*

$$\|\mathcal{N}(z,h)E_0 - T(z,h)E_0\|_{H_h^1(\Gamma)} \leq C_N h^{-s_0} h^N \|E_0\|_{L^2(\Gamma)} \tag{3.7}$$

with constants $C_N, s_0 > 0$, independent of E_0, h and z, and s_0 independent of N.

Proof of Theorem 1.1 in the case $z \in Z_2$. Consider the system

$$\begin{cases} \Big(D_\nu - \frac{1}{\gamma}\sqrt{z}\Big) E_{tan} - \Big(\operatorname{grad}_h E_{nor}\Big)\Big|_{tan} + \mathbf{i}h g_0(E_{tan}) = 0, \quad x \in \Gamma, \\ \operatorname{div}_h E_{tan} + \operatorname{div}_h \Big(E_{nor}\nu\Big) = 0, \quad x \in \Gamma, \end{cases} \tag{3.8}$$

where $\operatorname{div}_h F = \sum_{k=1}^3 D_{x_k} F_k$.

Take the scalar product $\langle , \rangle_{L^2(\Gamma)}$ in $L^2(\Gamma)$ of the first equation of (3.8) and E_{tan}. Applying Green formula, it easy to see that

$$-\operatorname{Re}\langle \operatorname{grad}_h E_{nor}\Big|_{tan}, E_{tan}\rangle_{L^2(\Gamma)} = -\operatorname{Re}\langle \operatorname{div}_h E_{tan}, E_{nor}\rangle_{L^2(\Gamma)}. \tag{3.9}$$

We claim that

$$\operatorname{Im}\langle g_0(E_{tan}), E_{tan}\rangle_{L^2(\Gamma)} = 0. \tag{3.10}$$

Let $E_{tan} = (w_1, w_2, w_3)$. Then

$$\langle g_0(E_{tan}), E_{tan}\rangle_{\mathbb{C}^3} = \sum_{k,j=1}^3 w_k \frac{\partial \nu_k}{\partial x_j}\overline{w_j} = \frac{1}{q}\sum_{k,j=1}^3 w_k \frac{\partial V_k}{\partial x_j}\overline{w_j} = \frac{1}{q}\langle Sw, w\rangle_{\mathbb{C}^3},$$

where $S := \{\frac{\partial V_k}{\partial x_j}\}_{k,j=1}^3$ with $V(x) = q(x)\nu(x)$, $q(x) > 0$ because $\sum_{k=1}^3 (\partial_{x_j} q) w_k \nu_k = 0$. Thus if the boundary is given locally by $x_3 = G(x_1, x_2)$, we choose $V(x) = (-\partial_{x_1} G, -\partial_{x_2} G, 1)$ and it is obvious that S is symmetric. Therefore $\operatorname{Im}\langle Sw, w\rangle_{\mathbb{C}^3} = 0$ and this proves the claim. Hence (3.10) implies

$$\operatorname{Re}[\mathbf{i}h\langle g_0(E_{tan}), E_{tan}\rangle_{L^2(\Gamma)}] = 0. \tag{3.11}$$

From the $L^2(\Gamma)$ scalar product of the second equation in (3.8) with E_{nor}, we obtain

$$\operatorname{Re}\langle \operatorname{div}_h E_{tan}, E_{nor}\rangle_{L^2(\Gamma)} + \operatorname{Re}\langle D_\nu E_{nor}, E_{nor}\rangle_{L^2(\Gamma)} = 0. \tag{3.12}$$

In fact,

$$\operatorname{div}_h(E_{nor}\nu) = D_\nu E_{nor} - \mathbf{i}hE_{nor}\operatorname{div}\nu$$

and $\operatorname{Im}\Big(\operatorname{div}\nu |E_{nor}|^2\Big) = 0$.

Taking together (3.9), (3.11) and (3.12), we conclude that

$$\operatorname{Re}\Big[\langle (D_\nu - \frac{\sqrt{z}}{\gamma})E_{tan}, E_{tan}\rangle_{L^2(\Gamma)} + \langle D_\nu E_{nor}\nu, E_{nor}\nu\rangle_{L^2(\Gamma)}\Big]$$

$$= \operatorname{Re}\Big\langle D_\nu E, E\Big\rangle_{L^2(\Gamma)} - \operatorname{Re}\langle \frac{\sqrt{z}}{\gamma} E_{tan}, E_{tan}\rangle_{L^2(\Gamma)} = 0.$$

Here we have used the fact that

$$\langle D_\nu E_{tan}, E_{nor}\nu\rangle_{\mathbb{C}^3} = D_\nu\Big(\langle E_{tan}, E_{nor}\nu\rangle_{\mathbb{C}^3}\Big) = 0.$$

Applying Proposition 3.1 with $E|_\Gamma = E_0$, yields

$$\Big|\operatorname{Re}\Big\langle T(z,h)E_0, E_0\Big\rangle_{L^2(\Gamma)} - \operatorname{Re}\Big\langle \frac{\sqrt{z}}{\gamma} E_{tan}, E_{tan}\Big\rangle_{L^2(\Gamma)}\Big| \leq C_N h^{-s_0} h^N \|E_0\|_{L^2(\Gamma)}. \tag{3.13}$$

For $z = -1$, as in Lemma 3.9 in [9] and Lemma 4.1 in [8], we have

$$|\operatorname{Re}\langle T(-1,h)E_0, E_0\rangle_{L^2(\Gamma)}| \leq C_N h^{-s_0+N} \|E_0\|^2_{L^2(\Gamma)} = 0.$$

Consequently, by using Taylor formula for the real-valued function

$$\operatorname{Re}\Big[\Big\langle T(z,h)E_0, E_0\Big\rangle_{L^2(\Gamma)} - \Big\langle \frac{\sqrt{z}}{\gamma} E_{tan}, E_{tan}\Big\rangle_{L^2(\Gamma)}\Big],$$

we get for every $N \in \mathbb{N}$ the estimate

$$\Big|\operatorname{Im}\Big[\Big\langle (\frac{\partial T}{\partial z}(z_t, h))E_0, E_0\Big\rangle_{L^2(\Gamma)} - \Big\langle \frac{\gamma_1}{2\sqrt{z_t}} E_{tan}, E_{tan}\Big\rangle_{L^2(\Gamma)}\Big]\Big|$$
$$\leq C_N \frac{h^{-s_0+N}}{|\operatorname{Im} z|} \|E_0\|^2_{L^2(\Gamma)}, \tag{3.14}$$

where $z_t = -1 + \mathbf{i}t\operatorname{Im} z,\ 0 < t < 1,\ \gamma_1 = \gamma^{-1}$.

According to Lemma 3.9 in [9], in (3.14) we can replace $\frac{\partial T}{\partial z}(z_t, h)$ by $Op_h(\frac{\partial \rho}{\partial z}(z_t))$ and this yields an error term bounded by $Ch\|E_0\|^2_{H_h^{-1}(\Gamma)}$. On the other hand,

$$\Big|\Big\langle Op_h(\frac{\partial\rho}{\partial z}(z_t))E_{tan}, E_{nor}\nu\Big\rangle_{L^2(\Gamma)} + \Big\langle Op_h(\frac{\partial\rho}{\partial z}(z_t))E_{nor}, E_{tan}\nu\Big\rangle_{L^2(\Gamma)}\Big|$$
$$\leq Ch\|E_0\|^2_{L^2(\Gamma)}$$

since the estimate (3.4) holds for $z \in Z_2$ with factor h and $\frac{\partial\rho}{\partial z}(z_t) \in S_0^{0,-1}$.

Thus the problem is reduced to a lower bound of

$$J := \Big|\mathrm{Im}\Big[\langle\Big(Op_h(\frac{\partial\rho}{\partial z}(z_t)) - \frac{\gamma_1}{2\sqrt{z}}\Big)E_{tan}, E_{tan}\rangle_{L^2(\Gamma)} + \langle Op_h(\frac{\partial\rho}{\partial z}(z_t))E_{nor}\nu, E_{nor}\nu\rangle_{L^2(\Gamma)}\Big]\Big|$$

$$\geq \Big|\mathrm{Im}\langle\Big(Op_h(\frac{\partial\rho}{\partial z}(z_t)) - \frac{\gamma_1}{2\sqrt{z}}\Big)E_{tan}, E_{tan}\rangle_{L^2(\Gamma)}\Big| - C_1\|E_{nor}\|^2_{L^2(\Gamma)}.$$

Since $\gamma_1(x) > 1$, $\forall x \in \Gamma$, applying the analysis of Section 4 in [8] for the scalar product involving E_{tan}, one deduces

$$\Big|\mathrm{Im}\Big\langle\Big(Op_h(\frac{\partial\rho}{\partial z}(z_t)) - \frac{\gamma_1}{2\sqrt{z}}\Big)E_{tan}, E_{tan}\Big\rangle_{L^2(\Gamma)}\Big| \geq \eta_1\|E_{tan}\|^2_{L^2(\Gamma)}, \quad \eta_1 > 0.$$

By using once more the estimate (3.4), for h small enough we obtain

$$J \geq \eta_1\Big(\|E_{tan}\|^2_{L^2(\Gamma)} + \|E_{nor}\|^2_{L^2(\Gamma)}\Big) - B_0 h\|E_{tan}\|^2_{L^2(\Gamma)} \geq \eta_2\|E_0\|^2_{L^2(\Gamma)}, \;\; 0 < \eta_2 < \eta_1.$$

Consequently, (3.14) yields

$$(\eta_2 - B_1 h)\|E_0\|^2_{L^2(\Gamma)} \leq C_N \frac{h^{-s_0+N}}{|\operatorname{Im} z|}\|E_0\|^2_{L^2(\Gamma)}$$

and for small h we conclude that for $z \in Z_2$ the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G_b lie in the region $\mathcal{R}_N$. This completes the analysis of the case $0 < \gamma(x) < 1$, $\forall x \in \Gamma$.

To study the case $\gamma(x) > 1$, $\forall x \in \Gamma$, we write the boundary condition in (1.1) as

$$\frac{1}{\gamma(x)}(\nu \wedge E_{tan}) - (\nu \wedge (\nu \wedge B_{tan})) = \frac{1}{\gamma(x)}(\nu \wedge E_{tan}) + B_{\tan} = 0.$$

Next

$$\nu \wedge E = \frac{1}{\sqrt{z}}\nu \wedge \frac{h}{\mathbf{i}}\operatorname{curl} B = -\frac{1}{\sqrt{z}}D_\nu B_{tan} + \frac{1}{\sqrt{z}}\Big[\Big(\operatorname{grad}_h B_{nor}\Big)\Big|_{tan} - \mathbf{i}hg_0(B_{tan})\Big]$$

and one obtains

$$\Big(D_\nu - \gamma(x)\sqrt{z}\Big)B_{tan} - \Big(\operatorname{grad}_h B_{nor}\Big)\Big|_{tan} + \mathbf{i}hg_0(B_{tan}) = 0, \; x \in \Gamma \tag{3.15}$$

which is the same as (2.1) with E_{tan}, E_{nor} replaced respectively by B_{tan}, B_{nor} and $\frac{1}{\gamma(x)}$ replaced by $\gamma(x) > 1$. We apply the operator $D_{y_1} - \gamma\sqrt{z}$ to the equation $\operatorname{div} B = 0$ and repeat without any change the above analysis concerning E_{tan}, E_{nor}. Thus the proof of Theorem 1.1 is complete. $\square$

Remark 3.2. The result of Theorem 1.1 holds for obstacles $K = \cup_{j=1}^J K_j$, where K_j, $j = 1, ..., J$ are open connected domains with C^∞ boundary and $K_i \cap K_j = \emptyset$, $i \neq j$. Let $\Gamma_j = \partial K_j$, $j = 1, ..., J$. In this case we may have $\gamma(x) < 1$ for some obstacles Γ_j and $\gamma(x) > 1$ for other ones. The proof extends with only minor modifications. The construction of the semi-classical parametrix in [9] is local and for the Dirichlet-to-Neumann map $\mathcal{N}_j(z,h)$ related to Γ_j we get the estimate

$$\|\mathcal{N}_j(z,h)(F) - Op_h(\rho + hb)F\|_{H^1_h(\Gamma_j)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}}\|F\|_{L^2(\Gamma_j)}.$$

The boundary condition in (1.1) is local and we can reduce the analysis to a fixed obstacle K_j. If $(E,B) \neq 0$ is an eigenfunction of G_b, our argument implies $E_{tan} = 0$ for $x \in \Gamma_j$ if $0 < \gamma(x) < 1$ on Γ_j and $B_{tan} = 0$ for $x \in \Gamma_j$ in the case $\gamma(x) > 1$ on Γ_j. By the boundary condition we get $E_{tan} = 0$ on Γ and this yields $E = B = 0$

since the Maxwell system with boundary condition $E_{tan} = 0$ has no eigenvalues in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

4. Appendix

In this Appendix, assume that $\gamma > 0$ is constant. Our purpose is to study the eigenvalues of G_b in case the obstacle is equal to the ball $B_3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$. Setting $\lambda = \mathbf{i}\mu$, $\operatorname{Im}\mu > 0$, an eigenfunction $(E, B) \neq 0$ of G_b satisfies

$$\operatorname{curl} E = -\mathbf{i}\mu B, \qquad \operatorname{curl} B = \mathbf{i}\mu E. \tag{4.1}$$

Replacing B by $H = -B$ yields for $(E, H) \in (H^2(|x| \leq 1))^6$,

$$\begin{cases} \operatorname{curl} E = \mathbf{i}\mu H, \qquad \operatorname{curl} H = -\mathbf{i}\mu E, \quad \text{for} \quad x \in B_3, \\ E_{tan} + \gamma(\nu \wedge H_{tan}) = 0, \quad \text{for} \quad x \in \mathbb{S}^2. \end{cases} \tag{4.2}$$

Expand $E(x), H(x)$ in the spherical functions $Y_n^m(\omega), n = 0, 1, 2, ..., |m| \leq n, \omega \in \mathbb{S}^2$ and the modified Hankel functions $h_n^{(1)}(z)$ of first kind. An application of Theorem 2.50 in [4] (in the notation of [4] it is necessary to replace ω by $\mu \in \mathbb{C} \setminus \{0\}$) says that the solution of the system (4.2) for $|x| = r = 1$ has the form

$$E_{tan}(\omega) = \sum_{n=1}^{\infty} \sum_{|m| \leq n} \Big[\alpha_n^m \Big(h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r)|_{r=1}\Big) U_n^m(\omega) + \beta_n^m h_n^{(1)}(\mu) V_n^m(\omega)\Big],$$

$$H_{tan}(\omega) = -\frac{1}{\mathbf{i}\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \Big[\beta_n^m \Big(h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r)|_{r=1}\Big) U_n^m(\omega) + \mu^2 \alpha_n^m h_n^{(1)}(\mu) V_n^m(\omega)\Big].$$

Here $U_n^m(\omega) = \frac{1}{\sqrt{n(n+1)}} \operatorname{grad}_{\mathbb{S}^2} Y_n^m(\omega)$ and $V_n^m(\omega) = \nu \wedge U_n^m(\omega)$ for $n \in \mathbb{N}, -n \leq m \leq n$ form a complete orthonormal basis in

$$L_t^2(\mathbb{S}^2) = \{u \in (L^2(\mathbb{S}^2))^3 : \langle \nu, u \rangle = 0 \text{ on } S^2\}.$$

To find a representation of $\nu \wedge H_{tan}$, observe that $\nu \wedge (\nu \wedge U_n^m) = -U_n^m$, so

$$(\nu \wedge H_{tan})(\omega) = -\frac{1}{\mathbf{i}\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \Big[\beta_n^m \Big(h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r)|_{r=1}\Big) V_n^m(\omega) - \mu^2 \alpha_n^m h_n^{(1)}(\mu) U_n^m(\omega)\Big]$$

and the boundary condition in (4.2) is satisfied if

$$\alpha_n^m \Big[h_n^{(1)}(\mu) + \frac{d}{dr}(h_n^{(1)}(\mu r))|_{r=1} - \gamma \mathbf{i}\mu h_n^{(1)}(\mu)\Big] = 0, \ \forall n \in \mathbb{N}, \ |m| \leq n, \tag{4.3}$$

$$-\frac{\beta_n^m \gamma}{\mathbf{i}\mu} \Big[h_n^{(1)}(\mu) + \frac{d}{dr}(h_n^{(1)}(\mu r))|_{r=1} - \frac{\mathbf{i}\mu}{\gamma} h_n^{(1)}(\mu)\Big] = 0, \ \forall n \in \mathbb{N}, \ |m| \leq n. \tag{4.4}$$

For $\gamma \equiv 1$, there are no eigenvalues.

Proposition 4.1. *For $\gamma \equiv 1$ the operator G_b has no eigenvalues in $\{\operatorname{Re} z < 0\}$.*

Proof. The functions $h_n^{(1)}(z)$ have the form (see for example [7])

$$h_n^{(1)}(x) \ = \ (-\mathbf{i})^{n+1} \frac{e^{\mathbf{i}x}}{x} \sum_{m=0}^{n} \frac{\mathbf{i}^m}{m!(2x)^m} \frac{(n+m)!}{(n-m)!} \ = \ (-\mathbf{i})^{n+1} \frac{e^{ix}}{x} R_n\Big(\frac{\mathbf{i}}{2x}\Big)$$

with

$$R_n(z) \ := \ \sum_{m=0}^{n} \frac{z^m}{m!} \frac{(n+m)!}{(n-m)!} \ = \ \sum_{m=0}^{n} a_m z^m.$$

Therefore the term in the brackets [...] in (4.3) becomes

$$(1-\gamma)\mathbf{i}\mu R_n\Big(\frac{\mathbf{i}}{2\mu}\Big) \;-\; \sum_{m=0}^{n} a_m m\Big(\frac{\mathbf{i}}{2\mu}\Big)^m .$$

Setting $w = \frac{\mathbf{i}}{2\mu}$, we must study for $\operatorname{Re} w > 0$ the roots of the equation

$$g_n(w) \;:=\; \frac{1-\gamma}{2w} R_n(w) + wR_n'(w) \;=\; 0\,. \tag{4.5}$$

For $\gamma = 1$ one obtains $R_n'(w) = 0$. A result of Macdonald says that the zeros of the function $h_n^{(1)}(z)$ lie in the half plane $\operatorname{Im} z < 0$ (see Theorem 8.2 in [7]), hence $R_n(w) \neq 0$ for $\operatorname{Re} w \geq 0$. By the theorem of Gauss-Lucas we deduce that the roots of $R_n'(w) = 0$ lie in the convex hull of the set of the roots of $R_n(w) = 0$, so $R_n'(w) \neq 0$ for $\operatorname{Re} w > 0$. Consequently, (4.3) and (4.4) are satisfied only for $\alpha_n^m = \beta_n^m = 0$ and $E_{tan} = 0$. This implies $E = H = 0$. $\square$

For the case $\gamma \neq 1$, there are an infinite number of real eigenvalues.

Proposition 4.2. *Assume that $\gamma \in \mathbb{R}^+ \setminus \{1\}$ is a constant. Then G_b has an infinite number of real eigenvalues. Let $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$. Then all real eigenvalues λ with exception of the eigenvalue*

$$\lambda_1 \;=\; -\frac{2}{(\gamma_0 - 1)\Big(1 + \sqrt{1 + \frac{4}{\gamma_0 - 1}}\Big)}\,. \tag{4.6}$$

satisfy the estimate

$$\lambda \;\leq\; -\frac{1}{\max\{(\gamma_0 - 1), \sqrt{\gamma_0 - 1}\}}\,. \tag{4.7}$$

Proof. Assume first that $\gamma > 1$. Then $q_n(w) = wg_n(w) = 0$ has at least one real root $w_0 > 0$. Indeed, $q_n(0) = \frac{1-\gamma}{2} < 0$, $q_n(w) \to +\infty$ as $w \to +\infty$. Choosing $\alpha_n^{m_0} \neq 0$ for an integer m_0, $|m_0| \leq n$ and taking all other coefficients α_n^m, β_n^m equal to 0, yields $E_{tan} \neq 0$ and G_b has an eigenfunction with eigenvalue $\lambda = -\frac{1}{2w_0} < 0$.

It is not excluded that $g_n(w)$ and $g_m(w)$ for $n \neq m$ have the same real positive root. If we assume that for $\operatorname{Re} w > 0$ the sequence of functions $\{g_n(w)\}_{n=1}^{\infty}$ has only a finite number of real roots $w_1, ..., w_N$, $w_j \in \mathbb{R}^+$, then there exists an infinite number of functions $g_{n_j}(w)$ having the same root which implies that we have an eigenvalue of G_b with infinite multiplicity. This is a contradiction, and the number of real eigenvalues of G_b is infinite.

It remains to establish the bound on the real eigenvalues. First, consider the case $n = 1$. Then one obtains the equation

$$\frac{2w^2}{2w+1} = \frac{\gamma - 1}{2}$$

which has a positive root $w_0 = \frac{1}{4}\Big(\gamma - 1 + \sqrt{(\gamma-1)^2 + 4(\gamma - 1)}\Big)$. This yields the λ_1 from (4.6)

Next examine the case $n \geq 2$. For a root $w_0 \in \mathbb{R}^+$ one has

$$w_0\Big(w_0 \frac{R_n'(w_0)}{R_n(w_0)}\Big) \;=\; \frac{\gamma - 1}{2}\,.$$

Case 1. $w_0 \geq \frac{1}{2\sqrt{3}}$. Then the inequality

$$\frac{\sum_{m=2}^n m a_m w_0^m + a_1 w_0}{\sum_{m=2}^n a_m w_0^m + a_1 w_0 + 1} \;\geq\; \frac{2\sum_{m=2}^n a_m w_0^m + a_1 w_0}{\sum_{m=2}^n a_m w_0^m + a_1 w_0 + 1} \;\geq 1$$

is satisfied since $a_2 = \frac{1}{2}(n+2)(n+1)n(n-1) \geq 12$. Consequently, $2w_0 \leq \gamma - 1$ and this implies that the eigenvalue $\lambda = -\frac{1}{2w_0}$ satisfies

$$\lambda \;<\; -\frac{1}{\gamma - 1}\,. \tag{4.8}$$

Case 2. $0 < w_0 \leq \frac{1}{2\sqrt{3}}$. Apply the inequality

$$\frac{\sum_{m=2}^n m a_m w_0^{m-1} + a_1}{w_0 \sum_{m=2}^n a_m w_0^{m-1} + a_1 w_0 + 1} \;\geq\; \frac{2\sum_{m=2}^n a_m w_0^{m-1} + a_1}{w_0 \sum_{m=2}^n a_m w_0^{m-1} + a_1 w_0 + 1} \;\geq\; 2$$

that is equivalent to

$$2\Big[(1 - w_0) S_0 - a_1 w_0\Big] + a_1 \geq 2$$

with $S_0 = \sum_{m=2}^n a_m w_0^{m-1}$. This inequality holds because

$$(1 - w_0) \sum_{m=2}^n a_m w_0^{m-1} - a_1 w_0 \;\geq\; (\frac{1}{2} a_2 - a_1) w_0\,, \quad a_1 = (n+1)n \geq 2,$$

and,

$$\frac{1}{2} a_2 - a_1 = \frac{1}{4}(n+2)(n+1)n(n-1) - (n+1)n = n(n+1)\Big[\frac{1}{4}(n+2)(n-1) - 1\Big] \geq 0.$$

Therefore,

$$2w_0^2 \;\leq\; w_0^2 \frac{\sum_{m=1}^n m a_m w_0^{m-1}}{\sum_{m=1}^n a_m w_0^m + 1} \;=\; \frac{\gamma - 1}{2}\,.$$

This easily yields

$$\lambda \leq -\frac{1}{\sqrt{(\gamma - 1)}}. \tag{4.9}$$

In the case $0 < \gamma < 1$ one has $1/\gamma > 1$ and we apply the above analysis to the equation (4.4). Setting $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$ and taking into account (4.8) and (4.9), we obtain the result. This completes the proof. □

Remark 4.3. Proposition 4.2 yields a more precise result than that in [1] since we prove the existence of an infinite number of real eigenvalues G_b for every $\gamma \in \mathbb{R}^+ \setminus \{1\}$. In the case $\gamma = \frac{1}{1+\epsilon}$, $\epsilon > 0$ the eigenvalue λ_1 has the form

$$\lambda_1 = \frac{1}{2}\Big(1 - \sqrt{1 + \frac{4}{\epsilon}}\Big)$$

and this result for small $\epsilon > 0$ has been obtained in [1]. Clearly, as $\gamma \to 1$ the real eigenvalues of G_b go to $-\infty$. .

It is easy to see that for $\gamma > 1$ the equation $g_n(w) = 0$ has no complex roots. Denote by

$$z_j, \quad \operatorname{Re} z_j < 0, \quad j = 1, ..., n,\ n \geq 1$$

the roots of $R_n(w) = 0$. Suppose that $g_n(w_0) = 0, n \geq 1$ with $\operatorname{Re} w_0 > 0, \operatorname{Im} w_0 \neq 0$. Then

$$\operatorname{Im}\Big[\frac{1-\gamma}{2w_0} + w_0 \sum_{j=1}^n \frac{1}{w_0 - z_j}\Big] = 0$$

and

$$-\frac{(1-\gamma)\operatorname{Im} w_0}{2|w_0|^2} + \operatorname{Re} w_0 \Big[-\sum_{j=1}^n \frac{\operatorname{Im} w_0}{|w_0 - z_j|^2} + \sum_{j=1}^n \frac{\operatorname{Im} z_j}{|w_0 - z_j|^2}\Big]$$
$$+ \operatorname{Im} w_0 \sum_{j=1}^n \frac{\operatorname{Re} w_0 - \operatorname{Re} z_j}{|w_0 - z_j|^2} \; = \; 0\,. \qquad (4.10)$$

On the other hand, if z_j with $\operatorname{Im} z_j \neq 0$ is a root of $R_n(w) = 0$, then $\bar{z}_j$ is also a root and

$$\frac{\operatorname{Im} z_j}{|w_0 - z_j|^2} - \frac{\operatorname{Im} z_j}{|w_0 - \bar{z}_j|^2} = \frac{\operatorname{Im} z_j}{|w_0 - z_j|^2 |w_0 - \bar{z}_j|^2}\Big(|w_0 - \bar{z}_j|^2 - |w_0 - z_j|^2\Big)$$
$$= \frac{4 \operatorname{Im} w_0 (\operatorname{Im} z_j)^2}{|w_0 - z_j|^2 |w_0 - \bar{z}_j|^2}\,.$$

Equation (4.10) becomes

$$\operatorname{Im} w_0 \Big[\frac{\gamma - 1}{2|w_0|^2} - \sum_{j=1}^n \frac{\operatorname{Re} z_j}{|w_0 - z_j|^2} + \sum_{\operatorname{Im} z_j > 0} \frac{4 \operatorname{Re} w_0 (\operatorname{Im} z_j)^2}{|w_0 - z_j|^2 |w_0 - \bar{z}_j|^2}\Big] \; = \; 0\,. \qquad (4.11)$$

The term in the brackets [...] is positive, and one concludes that $\operatorname{Im} w_0 = 0$.

Repeating the argument of the Appendix in [8], one can show that *for* $0 < \gamma < 1$ *the complex eigenvalues of* G_b *lie in the region*

$$\Big\{z \in \mathbb{C} :\ |\arg z - \pi| > \pi/4, \quad \operatorname{Re} z < 0\Big\}.$$

Remark 4.4. We do not know if there exist non real eigenvalues for B_3.

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Dipartimento di Matematica, Università di Pisa, Italia
E-mail address: colombini@dm.unipi.it

Institut de Mathématiques de Bordeaux, 351, Cours de la Libération, 33405 Talence, France
E-mail address: petkov@math.u-bordeaux.fr

Department of Mathematics, University of Michigan, USA
E-mail address: rauch@umich.edu